

Entire Dirichlet series with monotonous coefficients and logarithmic h -measure

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Abstract

Let F be an entire function represented by absolutely convergent for all $z \in \mathbb{C}$ Dirichlet series of the form $F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n}$, where a sequence (λ_n) such that $\lambda_n \in \mathbb{R}$ ($n \geq 0$), $\lambda_n \neq \lambda_k$ for any $n \neq k$ and $(\forall n \geq 0) : 0 \leq \lambda_n < \beta := \sup\{\lambda_j : j \geq 0\} \leq +\infty$. Let h be non-decrease positive continuous function on $[0, +\infty)$ and Φ increase positive continuous on $[0, +\infty)$ function.

In this paper we find the condition on (μ_n) and (λ_n) such that the relation $F(x + iy) = (1 + o(1))a_{\nu(x,F)}e^{(x+iy)\lambda_{\nu(x,F)}}$ holds as $x \rightarrow +\infty$ outside some set E of finite logarithmic h -measure uniformly in $y \in \mathbb{R}$.

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1 Introduction

Let \mathcal{L} be the class of positive continuous increasing functions on $[0; +\infty)$ and \mathcal{L}_+ the subclass of functions $\Phi \in \mathcal{L}$ such that $\Phi(t) \rightarrow +\infty$ ($t \rightarrow +\infty$). By φ we denote inverse function to $\Phi \in \mathcal{L}$.

Let \mathcal{D} be a class of entire (absolutely convergent in the whole complex plane \mathbb{C}) Dirichlet series of the form

$$F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n}, \quad z \in \mathbb{C}, \quad (1)$$

where a sequence (λ_n) such that $\lambda_n \in \mathbb{R}$ ($n \geq 0$), $\lambda_n \neq \lambda_k$ for any $n \neq k$ and

$$(\forall n \geq 0) : \quad 0 \leq \lambda_n < \beta := \sup\{\lambda_j : j \geq 0\} \leq +\infty. \quad (2)$$

For $F \in \mathcal{D}$ and $x \in \mathbb{R}$ we denote

$$M(x, F) = \sup\{|F(x + iy)| : y \in \mathbb{R}\}, \quad m(x, F) = \inf\{|F(x + iy)| : y \in \mathbb{R}\},$$

and by

$$\mu(x, F) = \max\{|a_n| e^{x\lambda_n} : n \geq 0\}, \quad \nu(x, F) = \max\{n : |a_n| e^{x\lambda_n} = \mu(x, F)\}$$

the maximal term and central index of series (1), respectively.

By \mathcal{D}_a we denote a subclass of Dirichlet series $F \in \mathcal{D}$ with a fixed sequence $a = (|a_n|)$, $|a_n| \searrow 0$ ($n_0 \leq n \rightarrow +\infty$), and for $\Phi \in \mathcal{L}$ by $\mathcal{D}_a(\Phi)$ subclass of functions $F \in \mathcal{D}_a$ such that $\ln \mu(x, F) \geq x\Phi(x)$ ($x \geq x_0$).

Let

$$\mu_n := -\ln |a_n| \quad (n \geq 0).$$

In paper [1] one can find such theorem:

Theorem A (O.B. Skaskiv, 1994 [1]). *For every entire function $F \in \mathcal{D}_a$ relation*

$$F(x + iy) = (1 + o(1)) a_{\nu(x, F)} e^{(x+iy)\lambda_{\nu(x, F)}} \quad (3)$$

holds as $x \rightarrow +\infty$ outside some set E of finite logarithmic measure, i.e. $\log\text{-meas}(E) := \int_E d \ln x < +\infty$, uniformly in $y \in \mathbb{R}$, if and only if

$$\sum_{n=n_0}^{+\infty} \frac{1}{\mu_{n+1} - \mu_n} < +\infty. \quad (4)$$

It is easy to see that relation (3) holds for $x \rightarrow +\infty$ ($x \notin E$) uniformly in $y \in \mathbb{R}$, if and only if

$$M(x, F) \sim \mu(x, F) \quad (x \rightarrow +\infty, x \notin E), \quad (5)$$

hence it follows

$$M(x, F) \sim m(x, F) \quad (x \rightarrow +\infty, x \notin E). \quad (6)$$

The finiteness of logarithmic measure of an exceptional set E in Theorem A is the sharp estimate. It follows from the following theorem.

Theorem B (Ya.Z. Stasyuk, 2008 [2]). *For every increasing sequence (μ_n) , such that condition (4) satisfies and for any function $h \in \mathcal{L}_+$ there exist an entire Dirichlet series $F \in \mathcal{D}_a$ with $|a_n| = \exp\{-\mu_n\}$, a set E and a constant $d > 0$ such that*

$$(\forall x \in E): \quad M(x, F) \geq (1 + d)\mu(x, F), \quad M(x, F) \geq (1 + d)m(x, F) \quad (7)$$

and

$$\text{h-log-meas}(E) := \int_{E \cap [1, +\infty)} h(x) d \ln x = +\infty.$$

Due to Theorem B the natural **question** arises: what conditions must satisfy the entire Dirichlet series $F \in \mathcal{D}_a$ such that relation (3) holds for $x \rightarrow +\infty$ outside some set E of finite logarithmic h-measure, i.e. $\text{h-log-meas}(E) < +\infty$? In this article we give the answer to this question. Our main result is the following.

Theorem 1.1. *Let (μ_n) be a sequence such that condition (4) holds, $h \in \mathcal{L}_+$, $\Phi \in \mathcal{L}$ and $F \in \mathcal{D}_a(\Phi)$. If*

$$(\forall b > 0): \quad \sum_{n=n_0}^{+\infty} h \left(\varphi(\lambda_n) \cdot \left(1 + \frac{b}{\mu_{n+1} - \mu_n} \right) \right) \frac{1}{\mu_{n+1} - \mu_n} < +\infty, \quad (8)$$

then the relation (3) holds as $x \rightarrow +\infty$ outside some set E of finite logarithmic h-measure uniformly in $y \in \mathbb{R}$.

The method of proof of Theorem 1.1 differs from the method of proofs corresponding statements in [1, 3] and is close to the methods of proofs from papers [4, 5, 6].

2 Proof of Theorem 1.1

We denote $\Delta_0 = 0$ and for $n \geq 1$

$$\Delta_n := \delta \cdot \sum_{j=0}^{n-1} (\mu_{j+1} - \mu_j) \sum_{m=j+1}^{+\infty} \left(\frac{1}{\mu_m - \mu_{m-1}} + \frac{1}{\mu_{m+1} - \mu_m} \right), \quad \delta > 0. \quad (9)$$

It easy to see that

$$\Delta_n \geq n\delta \quad (n \geq 0), \quad \Delta_n = o(\mu_n) \quad (n \rightarrow +\infty). \quad (10)$$

We put $b_n = e^{\lambda_n}$ and consider the Dirichlet series

$$f(s) = \sum_{n=0}^{+\infty} b_n e^{s\mu_n}, \quad \mu_n = -\ln |a_n|.$$

The condition $\sum_{n=0}^{+\infty} 1/(\mu_{n+1} - \mu_n) < +\infty$ implies $n^2 = o(\mu_n)$ ($n \rightarrow +\infty$) (see [1, 3]), thus $\ln n = o(\mu_n)$ ($n \rightarrow +\infty$). $F \in \mathcal{D}_a$ so $\lim_{n \rightarrow +\infty} \frac{-\ln |a_n|}{\lambda_n} = +\infty$. Therefore by Valiron's formulae for the abscissa of absolute convergence ([7, p.115]) we get

$$\sigma_{\text{abs}}(f) = \lim_{n \rightarrow +\infty} \frac{-\ln b_n}{\mu_n} = \lim_{n \rightarrow +\infty} \left(\frac{\ln |a_n|}{\lambda_n} \right)^{-1} = 0.$$

Now we consider Dirichlet series

$$f_q(s) = \sum_{n=0}^{+\infty} \frac{b_n}{\alpha_n^q} e^{s\mu_n}, \quad \alpha_n = e^{\Delta_n}, \quad q \in \mathbb{R}.$$

From the second relation in (10) and the condition $\sigma_{\text{abs}}(f) = 0$ it follows that

$$\sigma_{\text{abs}}(f_q) = \lim_{n \rightarrow +\infty} \frac{-\ln b_n + q\Delta_n}{\mu_n} = \sigma_{\text{abs}}(f) + q \cdot \lim_{n \rightarrow +\infty} \frac{\Delta_n}{\mu_n} = 0$$

for any $q \in \mathbb{R}$. Therefore the Dirichlet series of the form

$$f^*(s) = \sum_{n=0}^{+\infty} b_n e^{s\mu_n^*}, \quad \mu_n^* = \mu_n + \Delta_n,$$

is absolutely convergent in the whole half-plane $\Pi_0 := \{s : \operatorname{Re} s < 0\}$ and $\sigma_{\text{abs}}(f^*) = 0$. Indeed, for every fixed $s \in \Pi_0$ we have as $q = -\operatorname{Re} s$

$$\left| b_n e^{s\mu_n^*} \right| = \left| \frac{b_n}{\alpha_n^q} e^{s\mu_n} \right| \quad (\forall n \geq 0).$$

But $\sigma_{\text{abs}}(f_q) = 0$, thus $\sigma_{\text{abs}}(f^*) \geq 0$. In the other hand $b_n e^{s\mu_n^*} \Big|_{s=0} = e^{\lambda_n} \not\rightarrow 0$ ($n \rightarrow +\infty$), hence $\sigma_{\text{abs}}(f^*) = 0$.

From condition (2) (see proof of Lemma 2 in [1, p.121–122]) we get

$$\nu(x, f^*) \rightarrow +\infty \quad (x \rightarrow -0).$$

We need the following lemma (compare, for example, [4, 5, 6]).

Lemma 2.1. *For all $n \geq 0$ and $k \geq 1$ inequality*

$$\frac{\alpha_n}{\alpha_k} e^{\tau_k(\mu_n - \mu_k)} \leq e^{-\delta|n-k|} \quad (11)$$

holds, where

$$\tau_k := t_k + \frac{\delta}{\mu_k - \mu_{k-1}}, \quad t_k := \frac{\Delta_{k-1} - \Delta_k}{\mu_k - \mu_{k-1}}.$$

Proof of Lemma 2.1. We remark that

$$t_k = -\delta \cdot \sum_{m=k}^{+\infty} \left(\frac{1}{\mu_m - \mu_{m-1}} + \frac{1}{\mu_{m+1} - \mu_m} \right), \quad (12)$$

$$\tau_k = -2\delta \cdot \sum_{m=k+1}^{+\infty} \frac{1}{\mu_m - \mu_{m-1}}, \quad (13)$$

$$\tau_{k+1} - \tau_k = \frac{2\delta}{\mu_{k+1} - \mu_k} \quad (k \geq 1). \quad (14)$$

Since

$$\ln \alpha_n - \ln \alpha_{n-1} = \Delta_n - \Delta_{n-1} = -t_n(\mu_n - \mu_{n-1}),$$

for $n \geq k+1$ we have

$$\begin{aligned} \ln \frac{\alpha_n}{\alpha_k} + \tau_k(\mu_n - \mu_k) &= - \sum_{j=k+1}^n t_j(\mu_j - \mu_{j-1}) + \tau_k \sum_{j=k+1}^n (\mu_j - \mu_{j-1}) = \\ &= - \sum_{j=k+1}^n (t_j - \tau_k)(\mu_j - \mu_{j-1}) \leq - \sum_{j=k+1}^n (t_j - \tau_{j-1})(\mu_j - \mu_{j-1}) = \\ &= - \sum_{j=k+1}^n \delta = -(n - k) \cdot \delta. \end{aligned}$$

Similarly, for $n \leq k-1$ we obtain

$$\begin{aligned} \ln \frac{\alpha_n}{\alpha_k} + \tau_k(\mu_n - \mu_k) &= - \ln \frac{\alpha_k}{\alpha_n} - \tau_k(\mu_k - \mu_n) = \\ &= \sum_{j=n+1}^k t_j(\mu_j - \mu_{j-1}) - \tau_k \sum_{j=n+1}^k (\mu_j - \mu_{j-1}) = - \sum_{j=n+1}^k (\tau_k - t_j)(\mu_j - \mu_{j-1}) \leq \\ &\leq - \sum_{j=n+1}^k (\tau_j - t_j)(\mu_j - \mu_{j-1}) = - \sum_{j=n+1}^k \delta = -(k - n) \cdot \delta \end{aligned}$$

and Lemma 1 is proved. \square

We remark that from definitions of τ_k and t_k (see (13), (12)) and the condition $\sum_{k=0}^{+\infty} 1/(\mu_{k+1} - \mu_k) < +\infty$ it follows that there exists $k_0(\delta)$ such that

$$\tau_k \geq -1 \quad (k \geq k_0(\delta)), \quad \tau_k < 0 \quad (k \geq 1).$$

Let J be a set of the values of the central index $\nu(\sigma, f^*)$, i.e.

$$J = \{k \in \mathbb{N}: (\exists \sigma < 0)[\nu(\sigma, f^*) = k]\}.$$

Denote by (R_k) a sequence of the points of the springs of $\nu(\sigma, f^*)$, enumerate such that $\nu(\sigma, f^*) = k$ for $\sigma \in [R_k, R_{k+1})$ in the case $R_k < R_{k+1}$. Then for $\sigma \in [R_k, R_{k+1})$, $k \in J$ from Lemma 2.1 we have

$$b_n e^{\sigma \mu_n^*} \leq b_k e^{\sigma \mu_k^*} \iff \frac{b_n e^{\sigma \mu_n}}{b_k e^{\sigma \mu_k}} \leq \frac{\alpha_n^{|\sigma|}}{\alpha_k^{|\sigma|}} \leq e^{-|\sigma| \tau_k (\mu_n - \mu_k)} e^{-|\sigma| |n-k| \cdot \delta}$$

for all $n \geq 0$. Hence,

$$\frac{b_n e^{(\sigma + |\sigma| \tau_k) \mu_n}}{b_k e^{(\sigma + |\sigma| \tau_k) \mu_k}} \leq e^{-|\sigma| |n-k| \cdot \delta}$$

i.e., for all $k \in J$ and for every $\sigma^* \in [R_k(1 + |\tau_k|), R_{k+1}(1 + |\tau_k|))$

$$\frac{b_n e^{\sigma^* \mu_n}}{b_k e^{\sigma^* \mu_k}} \leq \exp \left\{ - \frac{|\sigma^*| |n-k| \cdot \delta}{1 + |\tau_k|} \right\}.$$

Thus, as $x = \frac{1}{|\sigma^*|} \in \left[-\frac{1}{R_k(1 + |\tau_k|)}, -\frac{1}{R_{k+1}(1 + |\tau_k|)} \right)$ we get

$$\frac{|a_n| e^{x \lambda_n}}{|a_k| e^{x \lambda_k}} \leq \exp \left\{ - \frac{|n-k| \cdot \delta}{1 + |\tau_k|} \right\}$$

for all $k \in J$, $n \geq 0$. Therefore,

$$\nu(x, F) = k, \quad \mu(x, F) = |a_k| e^{x \lambda_k}, \quad x \in \left[-\frac{1}{R_k(1 + |\tau_k|)}, -\frac{1}{R_{k+1}(1 + |\tau_k|)} \right). \quad (15)$$

Denote

$$E^*(\delta) := \bigcup_{k=k_0(\delta)}^{+\infty} \left[-\frac{1}{R_k(1 + |\tau_k|)}, -\frac{1}{R_{k+1}(1 + |\tau_k|)} \right), \quad E(\delta) := [0, +\infty) \setminus E^*(\delta).$$

Then for every $x > 0$, $x \notin E(\delta)$

$$\begin{aligned} & |F(x + iy) - a_{\nu(x, F)} e^{(x+iy) \lambda_{\nu(x, F)}}| \leq \\ & \leq \mu(x, F) \cdot \sum_{n \neq \nu(x, F)} \exp \left\{ - \frac{\delta \cdot |n - \nu(x, F)|}{1 + |\tau_{\nu(x, F)}|} \right\} \leq \frac{2e^{-\delta/2}}{1 - e^{-\delta/2}} \cdot \mu(x, F) \end{aligned} \quad (16)$$

because $1 + |\tau_{\nu(x, F)}| < 2$ ($x \in E^*(\delta)$). It remains to prove that the logarithmic h -measure of a set $E(\delta)$ is finite. Using

$$\begin{aligned} E(\delta) & \subset [0, x_0) \bigcup \left(\bigcup_{k=k_0(\delta)+1}^{+\infty} \left[-\frac{1}{R_k(1 + |\tau_{k-1}|)}, -\frac{1}{R_k(1 + |\tau_k|)} \right) \right), \\ x_0 & = -\frac{1}{R_{k_0(\delta)}(1 + |\tau_{k_0(\delta)-1}|)}, \end{aligned}$$

we obtain

$$\begin{aligned}
\text{h-log-meas}(E \cap [x_0, +\infty)) &= \sum_{k=k_0(\delta)+1}^{+\infty} \int_{\left[-\frac{1}{R_k(1+|\tau_{k-1}|)}, -\frac{1}{R_k(1+|\tau_k|)}\right)} h(x) d \ln x \leq \\
&\leq \sum_{k=k_0(\delta)+1}^{+\infty} h\left(-\frac{1}{R_k(1+|\tau_k|)}\right) \ln \frac{1+|\tau_{k-1}|}{1+|\tau_k|} = \\
&= \sum_{k=k_0(\delta)+1}^{+\infty} h\left(-\frac{1}{R_k(1+|\tau_k|)}\right) \ln \left(1 + \frac{|\tau_{k-1}| - |\tau_k|}{1+|\tau_k|}\right) \leq \\
&\leq \cdot \sum_{k=k_0(\delta)}^{+\infty} h\left(-\frac{1}{R_{k+1}(1+|\tau_{k+1}|)}\right) (|\tau_k| - |\tau_{k+1}|),
\end{aligned}$$

Hence, using equality (14) we have

$$\text{h-log-meas}(E \cap [x_0, +\infty)) \leq 2\delta \cdot \sum_{k=k_0(\delta)}^{+\infty} h\left(-\frac{1}{R_{k+1}(1+|\tau_{k+1}|)}\right) \frac{1}{\mu_{k+1} - \mu_k}. \quad (17)$$

The condition $F \in \mathcal{D}_a(\Phi)$ implies

$$x\Phi(x) \leq \ln \mu(x, F) = -\mu_{\nu(x-0, F)} + x\lambda_{\nu(x-0, F)} \leq x\lambda_{\nu(x-0, F)} \quad (x \geq x_1 > 0),$$

therefore

$$x \leq \varphi(\lambda_{\nu(x-0, F)}) \quad (x \geq x_1 > 0). \quad (18)$$

Denote $\theta_k := -\frac{1}{R_{k+1}(1+|\tau_k|)}$. By (15) we have $\nu(\theta_k - 0) = k$, thus from (14) and (18) it follows

$$\begin{aligned}
-\frac{1}{R_{k+1}(1+|\tau_{k+1}|)} &= \theta_k \cdot \frac{1+|\tau_k|}{1+|\tau_{k+1}|} = \theta_k \cdot \left(1 + \frac{|\tau_k| - |\tau_{k+1}|}{1+|\tau_{k+1}|}\right) \leq \\
&\leq \theta_k \cdot \left(1 + \frac{2\delta}{\mu_{k+1} - \mu_k}\right) \leq \varphi(\lambda_k) \cdot \left(1 + \frac{2\delta}{\mu_{k+1} - \mu_k}\right)
\end{aligned} \quad (19)$$

for all $k \geq k_1(\delta)$. Using inequality (19) to inequality (17), we get

$$\begin{aligned}
&\text{h-log-meas}(E(\delta) \cap [x_0, +\infty)) \leq \\
&\leq 2\delta \cdot \sum_{k=k_0(\delta)}^{k_2(\delta)-1} h\left(-\frac{1}{R_{k+1}(1+|\tau_{k+1}|)}\right) \frac{1}{\mu_{k+1} - \mu_k} + \\
&+ 2\delta \cdot \sum_{k=k_2(\delta)}^{+\infty} h\left(\varphi(\lambda_k) \cdot \left(1 + \frac{2\delta}{\mu_{k+1} - \mu_k}\right)\right) \frac{1}{\mu_{k+1} - \mu_k} := K(\delta) < +\infty, \quad (20)
\end{aligned}$$

where $k_2(\delta) = \max\{k_0(\delta), k_1(\delta)\}$. Relation (20) implies that

$$(\forall \delta > 0): \quad \text{h-log-meas}(E(\delta) \cap [x, +\infty)) = o(1) \quad (x \rightarrow +\infty).$$

We put now $\delta_n = n, \varepsilon_n = 2^{-n}$ ($n \geq 1$). Then for every $n \geq 1$ there exists $x_n \geq x_0$ such that

$$\text{h-log-meas}(E(\delta_n) \cap [x_n, +\infty)) \leq \varepsilon_n.$$

Without loss of generality we may assume that $x_n < x_{n+1}$ ($n \geq 1$). Denote $E = \bigcup_{n=1}^{+\infty} (E(\delta_n) \cap [x_n, x_{n+1}))$. Define a function $\gamma: [x_1, +\infty) \rightarrow [0, +\infty)$ by equality $\gamma(x) = \frac{2e^{-\delta_n/2}}{1 - e^{-\delta_n/2}}$ for $x \in [x_n, x_{n+1})$. Then from inequality (16) it follows

$$|F(x + iy) - a_{\nu(x, F)} e^{(x+iy)\lambda_{\nu(x, F)}}| \leq \gamma(x) \cdot \mu(x, F) \quad (21)$$

for all $x \in [x_1, +\infty) \setminus E$ uniformly in $y \in \mathbb{R}$. But $\gamma(x) \rightarrow 0$ ($x \rightarrow +\infty$) and

$$\text{h-log-meas}(E \cap [x_1, +\infty)) \leq \sum_{n=1}^{+\infty} \text{h-log-meas}(E(\delta_n) \cap [x_n, x_{n+1})) \leq \sum_{n=1}^{+\infty} \varepsilon_n = 1.$$

Thus, $\text{h-log-meas}(E) < +\infty$.

3 Consequences and concluding remarks

Remark 3.1. Since $\lambda_n/\mu_n \rightarrow 0$ ($n \rightarrow +\infty$), we have $\lambda_n < \mu_n$ for all n large enough. So λ_n one can replace with μ_n in (8).

In the case $\beta = \sup\{\lambda_j : j \geq 0\} = +\infty$ condition (8) of Theorem 1.1 can be written in a simpler form.

Let $\Phi \in \mathcal{L}_+$ and $\mathcal{D}_a^*(\Phi) := \bigcup_{\rho>0} \mathcal{D}_a(\Phi_\rho)$, $\Phi_\rho(x) := \rho \cdot \Phi(x\rho)$.

Theorem 3.1. Let (μ_n) be a sequence such that condition (4) holds, $h \in \mathcal{L}_+$, $\Phi \in \mathcal{L}_+$ and $F \in \mathcal{D}_a^*(\Phi)$. If

$$(\forall b > 0): \quad \sum_{n=n_0}^{+\infty} \frac{h(b\varphi(b\lambda_n))}{\mu_{n+1} - \mu_n} < +\infty, \quad (22)$$

then relation (3) holds as $x \rightarrow +\infty$ outside some set E of finite logarithmic h -measure uniformly in $y \in \mathbb{R}$.

Remark 3.2. In the case $\Phi(x) = e^x/x$ we obtain that $\mathcal{D}_a^*(\Phi)$ is the class Dirichlet series of nonzero lower R-order

$$\lim_{x \rightarrow +\infty} \frac{\ln \ln M(x, F)}{x} := \rho_R[F] \in (0, +\infty]$$

and condition (22) from condition

$$(\forall b > 0): \quad \sum_{n=n_0}^{+\infty} \frac{h(b \ln \lambda_n)}{\mu_{n+1} - \mu_n} < +\infty \quad (23)$$

follows.

Example 3.2. Well known, if $\ln n = o(\lambda_n \ln \lambda_n)$ and $\ln \lambda_{n+1} \sim \ln \lambda_n$ ($n \rightarrow +\infty$) then for the function $F \in D$ with coefficients

$$|a_n| = \exp \left\{ -\frac{1}{\rho} \lambda_n \ln \lambda_n \right\}$$

we have $\rho_R[F] = \rho$. Thus condition (23) follows from the condition

$$(\forall b > 0): \quad \sum_{n=n_0}^{+\infty} \frac{h(b \ln \lambda_n)}{\lambda_{n+1} \ln \lambda_{n+1} - \lambda_n \ln \lambda_n} < +\infty.$$

Question 3.1. Is the description of exceptional sets in our Theorems 1.1 and 3.1 the best possible?

Question 3.2. Are conditions (8) and (22) in our Theorems 1.1 and 3.1 necessary?

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